Last time: Existence & uniqueness
- Continuous dependence on ICs & parameters
- Lipschitz continuity

Today: Sensitivity w.r.t parameters
- Sensitivity equations
- Lyapunov-based stability (if time permits)

\[
\dot{x} = f(x(t, \mu(t)), \mu(t))
\]

- Assume \( f \) is continuous w.r.t both \( x \) and \( \mu \); \( f \) is continuously differentiable.
  (i.e. solution exists - at least on a finite time interval)
- \( \mu \) \in \mathbb{R}^m \), be a fixed vector of parameters

What happens if we perturb \( \mu \)?

Differentiate \( f \) w.r.t vector of parameters \( \mu \) & look at resulting ODE

\[
\begin{align*}
\dot{x}(t, \mu) &= x(t, \mu) + \frac{\partial x}{\partial \mu}(t, \mu) \cdot (\mu - \mu) + \text{h.o.t.} \\
\mu(t) &= \frac{\partial x}{\partial \mu}(t, \mu) \cdot S(t) - \text{"sensitivity" matrix} \\
S(t) &= \frac{\partial x}{\partial \mu}(t, \mu) \\
\end{align*}
\]

Can figure out what solution does w.r.t \( \mu \) by looking at sensitivity matrix

Objective: Find the eqn that governs the evolution of:

\[
S(t) = \left. \frac{\partial x}{\partial \mu}(t, \mu) \right|_{\mu} = \kappa_{\mu}(t, \mu)
\]

Plan of action: Differentiate w.r.t \( \mu \) first, then w.r.t. time

\[
\kappa_{\mu}(t, \mu) = \frac{\partial x}{\partial \mu} + \int_{t_0}^{t} \left( \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \mu} + \frac{\partial f}{\partial \mu} \right) \, dt
\]

\[
\kappa_{\mu}(t, \mu) = \int_{t_0}^{t} \left[ \frac{\partial f}{\partial x}(x(\tau, \mu), \mu) \cdot \kappa_{\mu}(\tau, \mu) + f_{\mu}(x(\tau, \mu), \mu, \tau) \right] \, d\tau \\
\forall \mu
\]

- Evaluate at \( \mu = \bar{\mu} \):

\[
\kappa_{\mu}(t, \bar{\mu}) = \int_{t_0}^{t} \left[ \frac{\partial f}{\partial x}(x(\tau, \bar{\mu}), \bar{\mu}) \cdot S(\tau) + f_{\mu}(x(\tau, \bar{\mu}), \bar{\mu}, \tau) \right] \, d\tau
\]

- Differentiate wrt. time:

\[
\dot{S}(t) = A(t) \cdot S(t) + B(t)
\]
where $A(t) = \frac{\partial f(x(t), \bar{\mu}, t)}{\partial x}$ and $B(t) = \frac{\partial f(x(t), \bar{\mu}, t)}{\partial \mu}$

Note: both matrices depend on solution $x(t, \bar{\mu})$

$$\dot{x} = f(x, \bar{\mu}, t)$$

$S' = eA(t) S(t) + B(t)$ \downarrow \text{one-way coupling}

$\rightarrow \text{simulate } \rightarrow \text{difficult to derive analytical sol.}$

$eA(t) & B(t)$ are functions of $x(t, \bar{\mu})$

$\rightarrow$ Eq (1): Fold bifurcation: $\dot{x} = x^2 + \mu$

$f(x, \mu) = x^2 + \mu$

$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial \mu} = 1$

$\dot{x} = x^2 + \mu, \quad x(0) = x_0$

$S = 2x(t) \dot{S} + 1; \quad S(0) = 0$

$\rightarrow$ Fixed trajectory that starts at $x_0$ for fixed $\bar{\mu}$

$\rightarrow$ Eq (2): (Khalil Eq. 3.17) $\dot{x}_1 = x_2$

$\dot{x}_2 = -c \sin(x_1) - [a + b \cos(x_1)] x_1 \quad = f_1$

$\mu = [a, b, c]^T \in \mathbb{R}^3$

$x(t) \in \mathbb{R}^2$

$S = \begin{bmatrix}
\frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} & \frac{\partial x_1}{\partial c} \\
\frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} & \frac{\partial x_2}{\partial c}
\end{bmatrix}\mu = \bar{\mu}$

$\frac{\partial f}{\partial x} = \begin{bmatrix}
\frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\
\frac{\partial f}{\partial x_4} & \frac{\partial f}{\partial x_5} & \frac{\partial f}{\partial x_6}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-c \cos(x_1) + b x_1 \sin(x_1) & -(a+b \cos(x_1))
\end{bmatrix}$

$A(t) = \begin{bmatrix}
0 & 1 \\
-c \cos(x(t)) & -1
\end{bmatrix}$

$B(t) = \begin{bmatrix}
0 & 0 \\
-x_1(t) & x_1(t) \cos(x(t)) & -\sin(x(t))
\end{bmatrix}$

$\rightarrow$ Lyapunov-based Stability

- stability w.r.t. initial conditions
- natural: BIBO w.r.t. inputs (undergrad)
- most commonly used notion of stability in science & engineering
• Consider: \( \mathbf{x} = f(\mathbf{x}) \rightarrow \) time-invariant, \( \mathbf{x}(t) \in \mathbb{R}^n \)

• Assume \( f(0) = 0 \Rightarrow \exists \mathbf{0} \) is an equ. pt.
  
  (w/o loss of generality) \( \Rightarrow \) if not \( \Rightarrow \) change coordinates
  
  \[
  \text{Consider } \mathbf{z} = f(\mathbf{z}) \text{ w/ } f(\mathbf{z}) = 0 \text{ w/ } \mathbf{z} \neq \mathbf{0}
  \]
  
  let \( \mathbf{z}(t) = \mathbf{x}(t) - \mathbf{z} \Rightarrow \dot{\mathbf{z}} = \mathbf{z} - \frac{\mathbf{z}}{f(\mathbf{z})} = f(\mathbf{z})
  \]
  
  Since \( f(\mathbf{z}) = 0 \Rightarrow \exists \mathbf{0} \) is an equ. pt.

• Stability: perturb equilibrium point w/ ICs and study qualitative behavior
  
  1) \( \mathbf{z} = 0 \) is stable (in the sense of Lyapunov)
  
  if \( \forall \varepsilon > 0, \exists \delta > 0 \)
  
  s.t. \( \| \mathbf{x}_0 \| < \delta \Rightarrow \| x(t, \mathbf{x}_0) \| < \varepsilon \)
  
  for all times
  
  if \( \mathbf{z} = 0 \) not an equ. pt: \( \| \mathbf{x}_0 - \mathbf{z} \| < \delta \Rightarrow \| x(t, \mathbf{x}_0) - \mathbf{z} \| < \varepsilon \)
  
  i.e. you start close, you stay close ...

  \[\begin{array}{c}
  \text{start at } \delta, \text{ stay w/in } \varepsilon \\
  \text{eg:} \text{ room temperature } \sim 69^\circ F
  \end{array}\]

  2) Unstable if it is not stable

  3) \underline{Locally asymptotically stable (LAS)}:
  
  - if it is stable
  
  - if \( \exists \delta > 0 \) s.t. \( \forall \| \mathbf{x}_0 \| < \delta \) \( \Rightarrow \lim_{t \to \infty} \| x(t, \mathbf{x}_0) \| = 0 \)

  (attractiveness)

  \( \Rightarrow \) doesn't say anything about stability

  4) \underline{Globally asymptotically stable (GAS)}:
  
  - if 3) holds for any \( \delta > 0 \)