HW#1

Show all work for full credit!

0. Reading Exercise: Read the CourseNotes for the last two weeks

1. Obtain a realization (the \( A, B, C, D \) matrices) of the following transfer functions

   (a) \( G(s) = \frac{s^2 + 2s + 3}{s^3 + 3s^2 + 4s + 5} \)

   (b) \( G(s) = \frac{s^3 + 3s^2 + 2s + 4}{2s^4 + 2s^3 + 3s^2 + 2} \)

   Provide the analog computer simulation schematic also (in terms of adders, integrators and amplifiers).

2. A system with input \( u(t) \) and output \( y(t) \) is given by

   \[
   \dot{y}(t) = u(t), \quad y(0) = 0.
   \]

   Prove that this system is linear, time-invariant, and causal (Assume that for all inputs are causal signals, that is, \( u(t) = 0 \) for \( t < 0 \)).

3. A model for the inverted pendulum on a cart shown in figure below ...

   ... is described by the following set of equations

   \[
   (M + m)\ddot{x} + ml\dot{\theta} \cos \theta = -b\dot{x} + m\dot{\theta}^2 \sin \theta + f
   \]

   \[
   (J + ml^2)\ddot{\theta} + ml\dot{x} \cos \theta = -mgl \sin \theta
   \]

   Assume \( M = m = 1 Kg, J = 1 Kg m^2, l = 1 m \) and \( b = 1 Kg/s \).

   (a) Obtain a four-dimensional nonlinear state-space representation with output \( y = x \), input \( u = f \), and states \([x_1 \ x_2 \ x_3 \ x_4] = [x \ \dot{x} \ \dot{\theta} \ \theta]\).

   (b) An equilibrium point of a system of the form \( \dot{x} = f(x, u) \) is obtained by solving for \( x \) that satisfies \( f(x, 0) = 0 \). Determine equilibrium points of this system (You can revise the notes on equilibrium points).

   (c) Linearize this system of equations around its trajectory about the equilibrium point \( x_{eq} = [0 \ \pi \ 0 \ 0] \) (when \( f(t) = 0 \)). Write it in state-space form.

   (d) Find the transfer function for the linear system obtained above.
4. **Discrete-time systems.** In class we have studied about continuous-time systems. We can make following analogous statements about the discrete-time systems. Read them and then solve the problem given below these statements.

(S1.) A nonlinear discrete-time system is described by a model of the form

\[
\begin{align*}
  x[k+1] &= f(x[k], u[k]),
  x(0) &= x_0 \\
  y(k) &= h(x[k], u[k]),
\end{align*}
\]

where \( k \in \mathbb{N} \) denotes the discrete time. A similar linearization scheme (as in the continuous time case) would lead to a linearized model

\[
\begin{align*}
  \tilde{x}[k+1] &= A[k] \tilde{x}[k] + B[k] \tilde{u}[k], \\
  \tilde{y}[k] &= C[k] \tilde{x}[k] + D[k] \tilde{u}[k]
\end{align*}
\]

where, again, the \( \{A[k], B[k], C[k], D[k]\} \) are obtained by evaluating the Jacobian matrices about the nominal state \( \{x_{nom}[k], u_{nom}[k]\} \).

(S2.) An *equilibrium point* of a discrete-time system of the form \( x[k+1] = f(x[k], u[k]) \) is obtained by solving for \( x \) that satisfies \( f(x, 0) = x \).

Consider the logistics equation, which is a basic model used in studying population dynamics,

\[
x[k+1] = \alpha x[k] - \alpha x^2[k], x[0] = x_0, \alpha > 1
\]

(a) Compute the two equilibrium points of this system.

(b) Determine the linearized system approximation about each equilibrium point.

(c) Let \( \alpha = 1.5 \). Determine \( \tilde{x}[10] \) for each linearized system in terms of \( \tilde{x}[0] \). State about which equilibrium is the corresponding linearized system a better approximation of the nonlinear logistics equation.

5. Consider the quarter car model shown, \( r, y \) and \( z \) denote the road height, car position and the wheel position respectively. \( F \) is a force acting on the wheel. Obtain a state space realization of the quarter car model with input being \([r, F]^T\) and output being the car position \( y \). Obtain the transfer functions from the input to the output.