Problem 1 Let $S$ denote the set of all stable proper real-rational functions. Show that any element $G \in S$ can be decomposed as $G = G_{ap}G_{mp}$ where $G_{ap}$ and $G_{mp}$ are all-pass and minimum phase respectively. Also prove that such a decomposition is unique up to a sign change.

Problem 2: Is it true that for every $\delta > 1$ there exists an internally stabilizing controller such that $\|T\|_{H\infty} < \delta$.

Problem 3 Let $W_p$ be the tracking performance weight that is chosen as a Butterworth filter with cutoff frequency of 1 rad/s. Plot $|W_p(z)|, z = 0.1 + j\omega$, from $\omega = 0$ to a value of $\omega$ where $|W_p| < 0.01$. Repeat the same for abscissae of $z = 1 + j\omega$, and $z = 10 + j\omega$. Comment on the limitation imposed on the sensitivity transfer function $S$ if the plant had zero at $z$.

Problem 4 Let $P = \frac{4}{s^2 - s + 4}$. Suppose $K$ is an internally stabilizing controller in a unity negative feedback configuration such that $\|S\|_{H\infty} = 1.5$. Give a positive lower bound for $\max_{0 \leq \omega < 0.1} |S(j\omega)|$.

Problem 5 Define $\epsilon := \|W_pS\|_{H\infty}$ and $\delta := \|KS\|_{H\infty}$. As we have seen in class, $\epsilon$ and $\delta$ capture tracking performance and controller effort respectively. Show that for every $s_0$ in the right half complex plane

$$|W_p(s_0)| \leq \epsilon + |W_p(s_0)P(s_0)|\delta.$$ 

This implies that $\epsilon$ and $\delta$ cannot be simultaneously made small indicating a tradeoff has to be reached between tracking and controller effort objectives.

Problem 6: Let $\omega$ be a frequency such that $j\omega$ is not a pole of the plant $P$ in a unity negative feedback configuration. Suppose that $\epsilon := |S(j\omega)| < 1$. Derive a lower bound for $|K(j\omega)|$ where $K$ is a stabilizing controller. Conclusion: Good tracking at a particular frequency requires large controller gain at that frequency.

Problem 7 Suppose that the plant is given by

$$P = \frac{1}{s^2 - s + 4}.$$ 

Suppose the controller $K$ in a unity negative feedback configuration achieves the following

- Internal stability
- $|S(j\omega)| \leq \epsilon$ for $0 \leq \omega < 0.1$.
- $|S(j\omega)| \leq 2$ for $0.1 \leq \omega < 5$. 

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• $|S(j\omega)| \leq 1$ for $5 \leq \omega < \infty$

Find a positive lower bound on the achievable $\epsilon$.

**Problem 8**

1. Prove that if a unity negative feedback system is stable, $z_j$, $j = 1,\ldots,N_z$ are rhp zeros of the plant, $p_i$, $i = 1,\ldots,N_p$ are the rhp poles of the plant and $\theta$ denotes the time delay in the plant, then

$$M_S := \|S\|_{\mathcal{H}_\infty} \geq \prod_{i=1}^{N_p} \frac{|z_j + p_i|}{|z_j - p_i|} =: M_{zp}, \text{ for all } j$$

and

$$M_T := \|T\|_{\mathcal{H}_\infty} \geq \prod_{j=1}^{N_z} \frac{|z_j + p_i|}{|z_j - p_i|} |e^{p_i \theta}| =: M_{pz}, \text{ for all } i$$

with $S$ and $T$ denoting the sensitivity and complimentary sensitivity closed-loop transfer functions.

2. Prove that

$$\|S\|_{\mathcal{H}_\infty} \geq \|T\|_{\mathcal{H}_\infty} - 1.$$ 

**Problem 9** Consider the plant

$$G(s) = 10 \frac{s - 2}{s^2 - 2s + 5}.$$ 

Show that $\|S\|_{\mathcal{H}_\infty} \geq 2.6$ and $\|T\|_{\mathcal{H}_\infty} \geq 2.6$.

**Problem 10** The "minimum and stable version," $G_{ms}$ of any transfer function $G$, is defined by

$$G_{ms} = \prod_i \frac{s - p_i}{s + p_i} G(s) \prod_j \frac{s + z_j}{s - z_j}.$$ 

We further denote

$$G_s(s) = \prod_i \frac{s - p_i}{s + p_i} G(s) \text{ and } G_m(s) = G(s) \prod_j \frac{s + z_j}{s - z_j}.$$ 

Let $VT$ be a weighted complimentary sensitivity transfer function with $V$ and $T$ being the weight and complimentary sensitivity transfer function respectively.

Let $V_{ms}$ be the "stable and minimum phase" version of $V$. Suppose $p$ is a rhp pole of the plant $G$. Show that if the closed-loop systems is stable then
1. \( \|VT\|_{\mathcal{H}_\infty} \geq |V_{ms}(p)|\prod_{j=1}^{N_s} |\zeta_j + p| e^{j\theta}. \)

2. \( \|KS\|_{\mathcal{H}_\infty} \geq |G_s(p)|^{-1}. \)

**Problem 11**

Consider the weight

\[
wp(s) = \frac{s + M\omega_B}{s + fM^2\omega_B} \cdot \left( \frac{s + fM\omega_B}{s + M\omega_B} \right) \cdot \left( \frac{10s + 1}{100s + 1} \right)
\]

with \( f > 1 \). This weight is the same as the weight \( \frac{s + M\omega_B}{s + \omega_B^A} \) with \( A = 0 \) and that it approaches 1 at high frequencies where \( f \) gives a frequency range over which a peak is allowed. Plot the weight for \( f = 10 \) and \( M = 2 \). Derive an upper bound on \( \omega_B \) in the case with \( f = 10 \) and \( M = 2 \).

**Problem 12**

Consider the weight

\[
w_p = \frac{1}{M} + \left( \frac{\omega_B}{s} \right)^n
\]

on the sensitivity transfer function \( S \). The weight requires the magnitude of the bode plot of \( |S| \) to have a slope of \( n \) at low frequencies and requires its low frequency asymptote to cross 1 at the frequency \( \omega_B \). Derive an upper bound on \( \omega_B \) when the plant has a rhp zero at \( z \). Show that bound becomes smaller than \( |z| \) as \( n \to \infty \).

**Problem 13** [Limitation due to rhp zero]

Consider the case of a plant with a rhp zero at \( z \). The weight on the sensitivity transfer function is chosen as

\[
w_p(s) = \frac{1000s}{\omega_B} + \frac{1}{M}\left( \frac{s}{M\omega_B} + 1 \right) \cdot \left( \frac{10s}{\omega_B} + 1 \right) \cdot \left( \frac{100s}{\omega_B} + 1 \right)
\]

This weight is close to \( 1/M \) at low and high frequencies, has a maximum close to \( 10/M \) and intermediate frequencies and a asymptote that crosses 1 at frequencies \( \omega_BL \omega_B/1000 \) and \( \omega_BH = \omega_B \). Thus we need good tracking \( (|S| < 1) \) in the frequency range between \( \omega_BL \) and \( \omega_BH \).

1. Sketch \( \frac{1}{|w_p|} \)

2. Show that \( |z| \) cannot be in the region where good tracking is needed and that we can achieve good tracking at frequencies either below \( \frac{z}{2} \) or above \( 2z \). To see this select \( M = 2 \) and evaluate \( w_p(z) \) at various values of \( \omega_B = k\omega, \) \( k = 0.1, 0.5, 1, 10, 100, 1000, 2000. \)