Chapter 14: Redundant Arithmetic

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- A non-redundant radix-r number has digits from the set{0, 1, ... , r - 1} and all numbers can be represented in a unique way.
- A radix-r redundant signed-digit number system is based on digit set $S \equiv \{-\beta, -(\beta - 1), \ldots, -1, 0, 1, \ldots, \alpha\}$, where, $1 \leq \beta, \alpha \leq r - 1$.
- The digit set S contains more than r values $\Rightarrow$ multiple representations for any number in signed digit format. Hence, the name redundant.
- A symmetric signed digit has $\alpha = \beta$.
- Carry-free addition is an attractive property of redundant signed-digit numbers. This allows most significant digit (msd) first redundant arithmetic, also called <u>on-line arithmetic</u>.

Redundant Number Representations

- A symmetric signed-digit representation uses the digit set $D_{<r.\alpha>} = \{-\alpha, \ldots, -1, 0, 1, \ldots, \alpha\}$, where r is the radix and α the largest digit in the set. A number in this representation is written as :

$$X_{<r.\alpha>} = x_{W-1}.x_{W-2}.x_{W-3}\ldots x_0 = \sum x_{W-1-i} r^i$$

The sign of the number is given by the sign of the most significant non-zero digit.

Digit Set $D_{<r.\alpha>}$	α	Redundancy Factor ρ
Incomplete	$< (r-1)/2$	$< ½$
Complete but non-redundant	$= (r-1)/2$	$= ½$
Redundant	$\geq \lceil r/2 \rceil$	$> ½$
Minimally redundant	$= \lceil r/2 \rceil$	$> ½$ and < 1
Maximally redundant	$= r-1$	$= 1$
Over-redundant	$> r-1$	> 1

Hybrid Radix-2 Addition

$$S_{<2.1>} = X_{<2.1>} + Y$$

where, $X_{<r.\alpha>} = x_{W-1}.x_{W-2}x_{W-3}...x_0$, $Y = y_{W-1}.y_{W-2}y_{W-3}...y_0$. The addition is carried out in two steps :

1. The 1st step is carried out in parallel for all the bit positions. An intermediate sum $p_i = x_i + y_i$ is computed, which lies in the range $\{\bar{1}, 0, 1, 2\}$. The addition is expressed as:

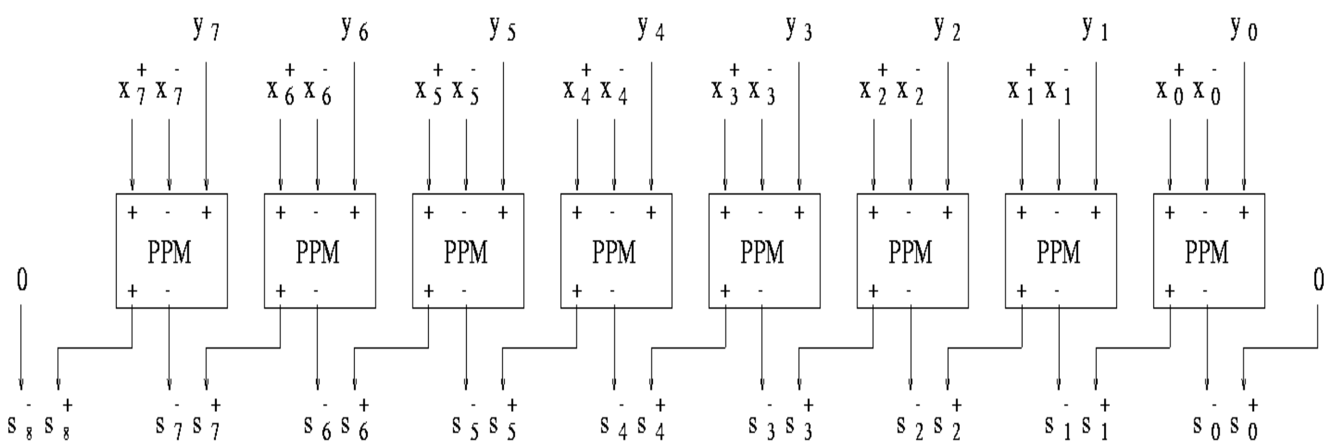
   $$x_i + y_i = 2t_i + u_i,$$

   where t_i is the transfer digit and has value 0 or 1, and is denoted as t_i^+; u_i is the interim sum and has value either 1 or 0 and is denoted as $-u_i^-$. t_{-1} is assigned the value of 0.

2. The sum digits s_i are formed as follows:

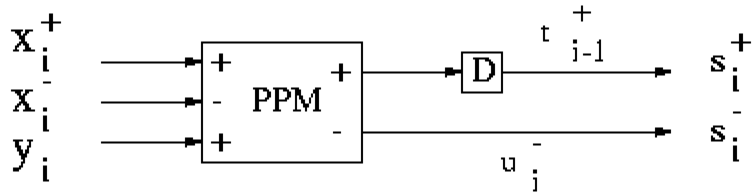
   $$s_i = t_{i-1}^+ - u_i^-$$

Digit	Radix 2 Digit Set	Binary Code
x_i	$\{\bar{1}, 0, 1\}$	$x_i^+ - x_i^-$
y_i	$\{0, 1\}$	y_i^+
$p_i = x_i + y_i$	$\{\bar{1}, 0, 1, 2\}$	$2t_i + u_i$
u_i	$\{\bar{1}, 0\}$	$-u_i^-$
t_i	$\{0, 1\}$	t_i^+
$s_i = u_i + t_{i-1}$	$\{\bar{1}, 0, 1\}$	$s_i^+ - s_i^-$

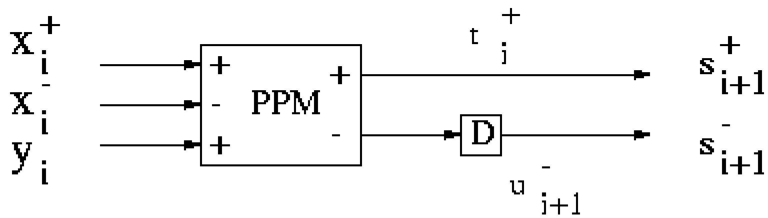


Eight-digit hybrid radix-2 adder

Digit-serial adder formed by folding



LSD-first adder



MSD-first adder

Hybrid Radix-2 Subtraction

$$S_{<2.1>} = X_{<2.1>} - Y$$

where, $X_{<r.\alpha>} = x_{W-1}.x_{W-2}x_{W-3}\ldots x_0$, $Y = y_{W-1}.y_{W-2}y_{W-3}\ldots y_0$. The addition is carried out in two steps :

1. The 1st step is carried out in parallel for all the bit positions. An intermediate difference $p_i = x_i - y_i$ is computed, which lies in the range $\{\bar{2}, \bar{1}, 0, 1\}$. The addition is expressed as:

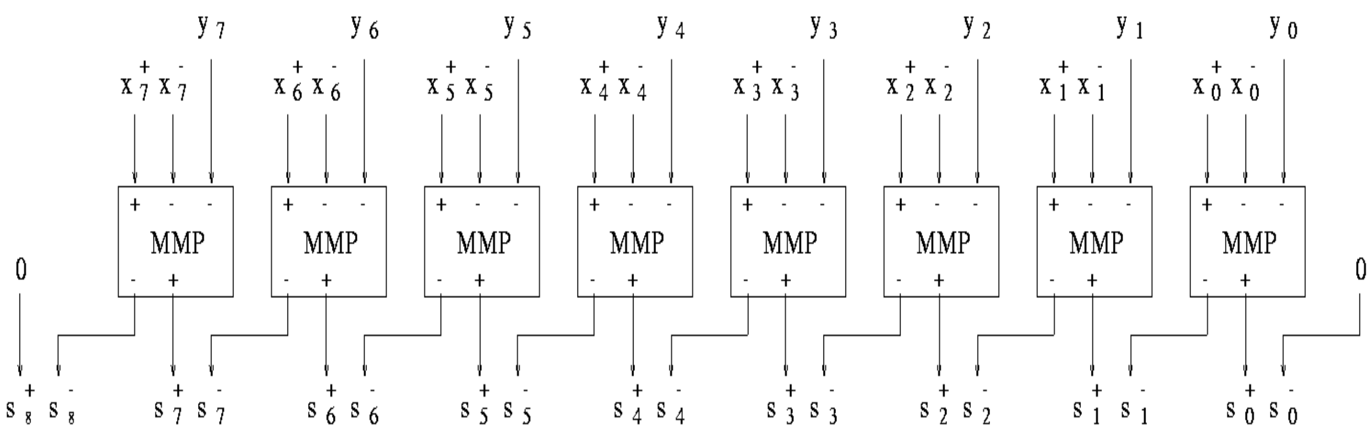
$$x_i - y_i = 2t_i + u_i,$$

where t_i is the transfer digit and has value 1 or 0, and is denoted as $-t_i^-$; u_i is the interim sum and has value either 0 or 1 and is denoted as u_i^+. t_{-1} is assigned the value of 0.

2. The sum digits s_i are formed as follows:

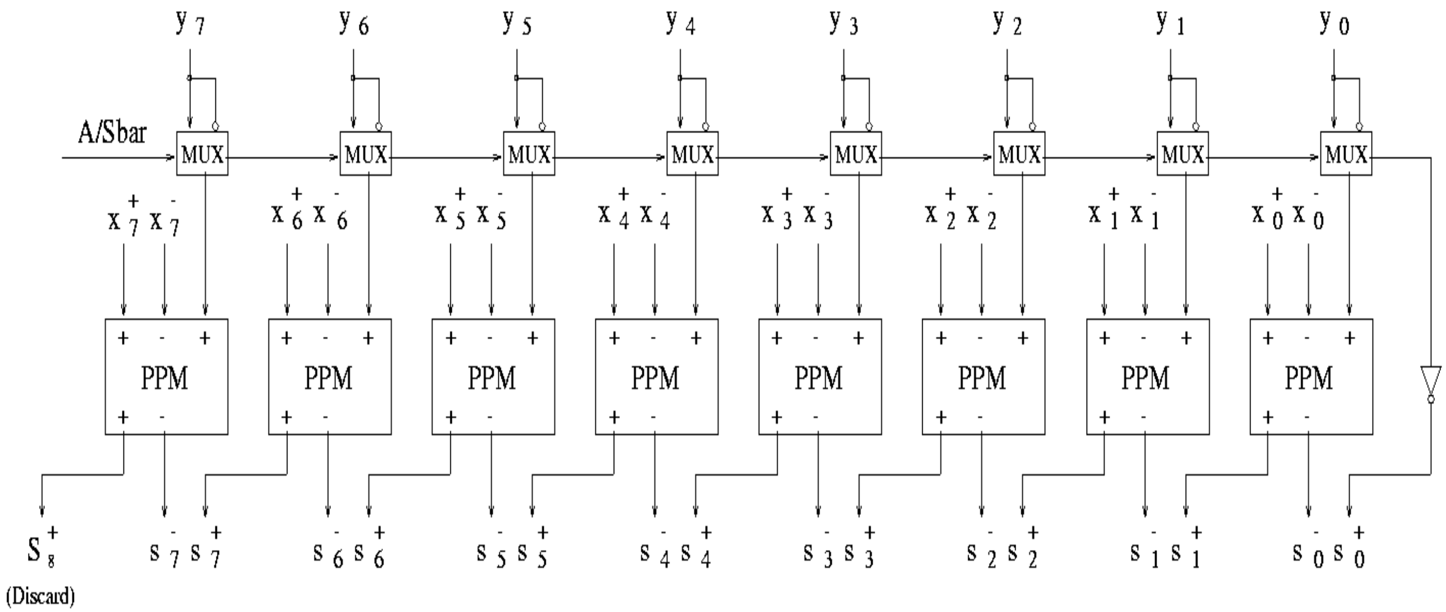
$$s_i = -t_{i-1}^- + u_i^+$$

Digit	Radix 2 Digit Set	Binary Code
x_i	$\{\bar{1}, 0, 1\}$	$x_i^+ - x_i^-$
y_i	$\{0, 1\}$	y_i^-
$p_i = x_i - y_i$	$\{\bar{2}, \bar{1}, 0, 1\}$	$2t_i + u_i$
u_i	$\{0, 1\}$	u_i^+
t_i	$\{\bar{1}, 0\}$	$-t_i^-$
$s_i = u_i + t_{i-1}$	$\{\bar{1}, 0, 1\}$	$s_i^+ - s_i^-$



Eight-digit hybrid radix-2 subtractor

Hybrid Radix-2 Addition/Subtraction



Hybrid radix-2 adder/subtractor ($A/\overline{S} = 1$ for addition and $A/\overline{S} = 0$ for subtraction)

- This is possible if one of the operands is in radix-r complement representation. Hybrid subtraction is carried out by hybrid addition where the 2's complement of the subtrahend is added to the minuend and the carry-out from the most significant position is discarded.

Signed Binary Digit (SBD) Addition/Subtraction

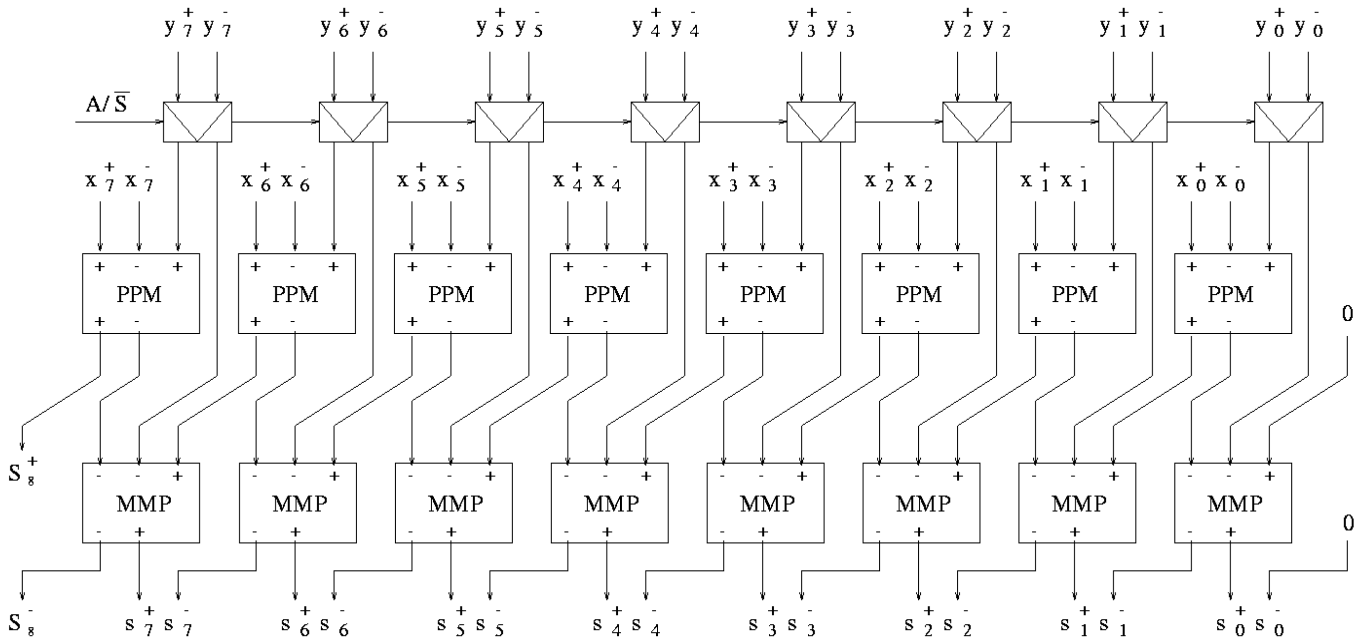
- $Y_{<r.\alpha>} = Y^{+} - Y^{-}$, is a signed digit number, where Y^{+} and Y^{-} are from the digit set $\{0, 1, \ldots, \alpha\}$.
- A signed digit number is thus subtraction of 2 unsigned conventional numbers.
- Signed addition is given by:

$$S_{<r.\alpha>} = X_{<r.\alpha>} + Y_{<r.\alpha>} = X_{<r.\alpha>} + Y^{+} - Y^{-},$$

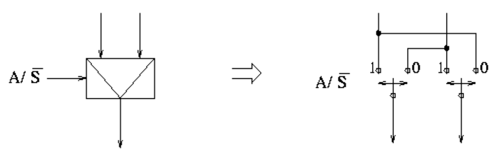
$$\Rightarrow S1_{<r.\alpha>} = X_{<r.\alpha>} + Y^{+},$$

$$S_{<r.\alpha>} = S1_{<r.\alpha>} - Y^{-}$$

- Digit serial SBD adders can be derived by folding the digit parallel adders in both lsd-first and msd-first modes.
- LSD-first adders have zero latency and msd-first adders have latency of 2 clock cycles.

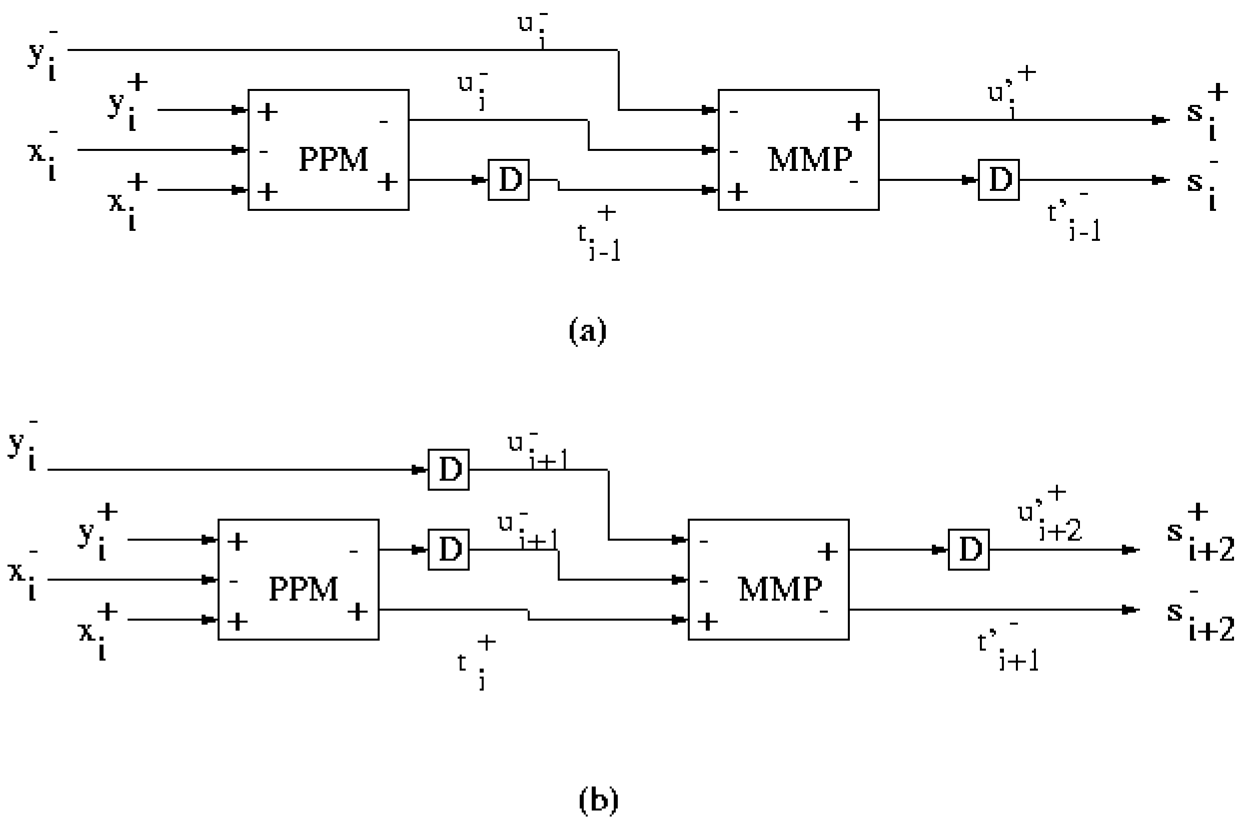


(a)



(b)

(a) Signed binary digit adder/subtractor
(b) Definition of the switching box



Digit serial SBD redundant adders. (a) LSD-first adder (b) msd-first adder

Maximally Redundant Hybrid Radix-4 Addition (MRHY4A)

- Maximally redundant numbers are based on digit set $D_{<4.3>}$.

$$S_{<4.3>} = X_{<4.3>} - Y_4$$

- The first step computes:

$$x_i + y_i = 4t_i + u_i$$

Replacing the respective binary codes from the table the following is obtained :

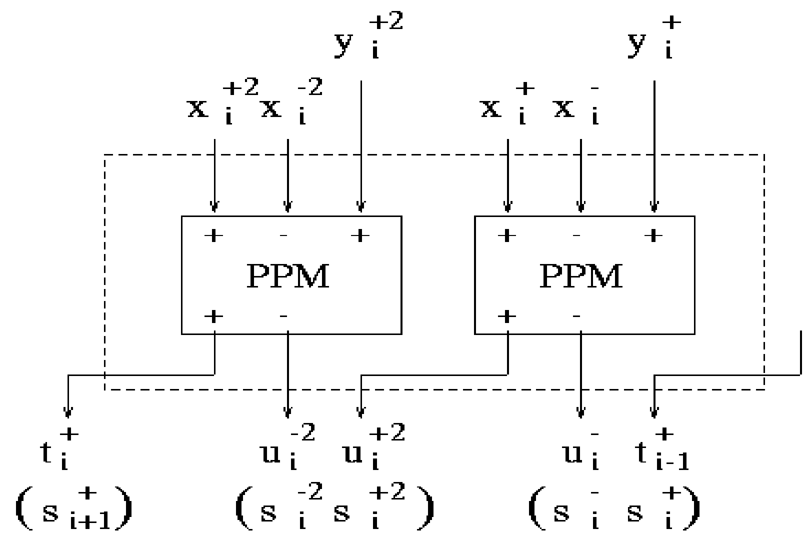
$$(2x_i^{+2} - 2x_i^{-2} + 2y_i^{+2}) + x_i^{+} - x_i^{-} + y_i^{+} = 4t_i^{+} + 2u_i^{+2} - 2u_i^{-2} - u_i^{-}$$

A MRHY4A cell consisting of two PPM adders is used to compute the above.

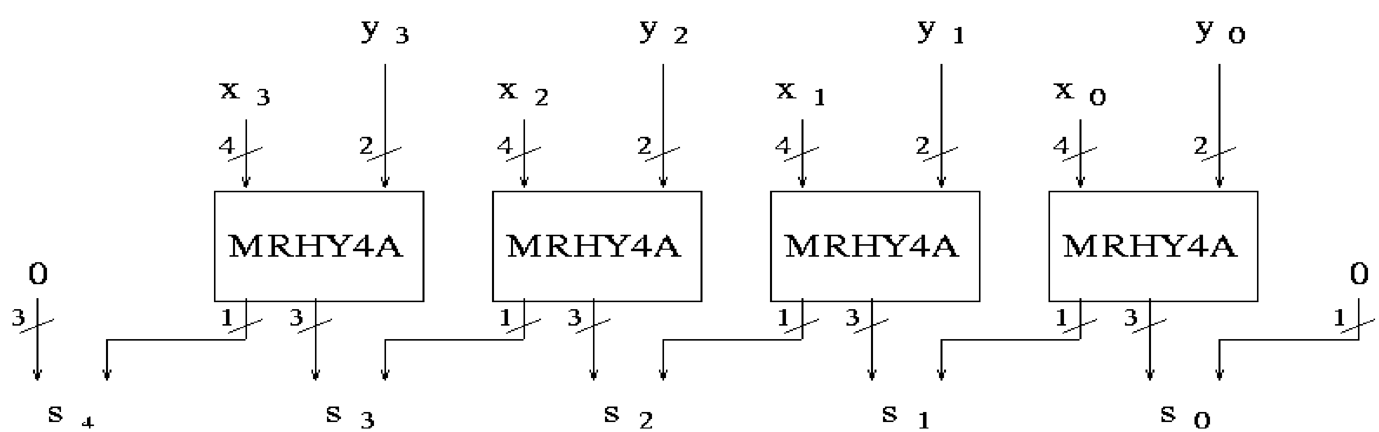
- Step 2 computes computes $s_i = t_{i-1} + u_i$. Replacing s_i, u_i, and t_{i-1} by corresponding binary codes leads to $s_i^{+2} = u_i^{+2}$, $s_i^{-2} = u_i^{-2}$, $s_i^{+}=t_{i-1}^{+}$ and $s_i^{-} = u_i^{-}$.

Digit	Radix 4 Digit Set	Binary Code
x_i	$\{\bar{3}, \bar{2}, \bar{1}, 0, 1, 2, 3\}$	$2x_i^{+2} – 2x_i^{-2} + x_i^{+} - x_i^{-}$
y_i	$\{0, 1, 2, 3\}$	$2y_i^{+2} + y_i^{+}$
$p_i = x_i + y_i$	$\{\bar{3}, \bar{2}, \bar{1}, 0, 1, 2, 3, 4, 5, 6\}$	$4t_i + u_i$
u_i	$\{\bar{3}, \bar{2}, \bar{1}, 0, 1, 2\}$	$2u_i^{+2} – 2u_i^{-2} - u_i^{-}$
t_i	$\{0, 1\}$	t_i^{+}
$s_i = u_i + t_{i-1}$	$\{\bar{3}, \bar{2}, \bar{1}, 0, 1, 2, 3\}$	$2s_i^{+2} - 2s_i^{-2} + s_i^{+} - s_i^{-}$

Digit sets involved in Maximally Redundant Hybrid Radix-4 Addition



MRHY4A adder cell



Four-digit MRHY4A

Minimally Redundant Hybrid Radix-4 Addition (mrHY4A)

- Minimally redundant numbers are based on digit set $D_{<4.2>}$.

$$S_{<4.2>} = X_{<4.2>} - Y_4$$

- The first step computes:

$$x_i + y_i = 4t_i + u_i$$

Replacing the respective binary codes from the table the following is obtained :

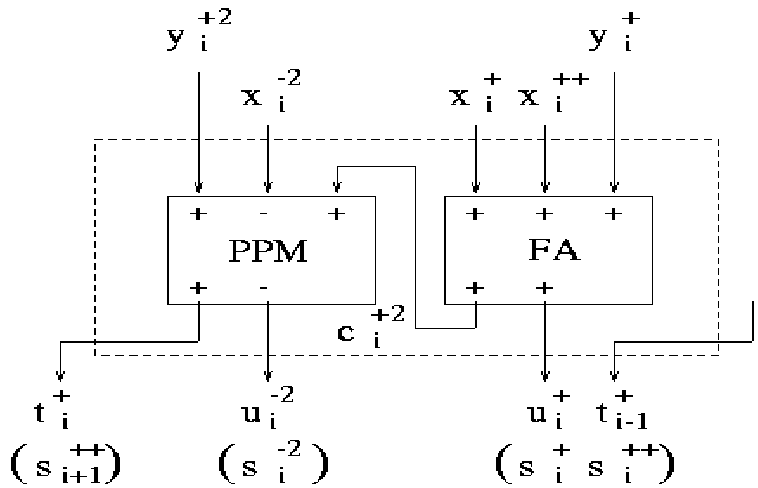
$$(-2x_i^{-2} + 2y_i^{+2}) + (x_i^{+} + x_i^{++} + y_i^{+}) = 4t_i^{+} - 2u_i^{-2} + u_i^{+}$$

A mrHY4A cell consisting of one PPM adder and a full adder is used to compute the above.

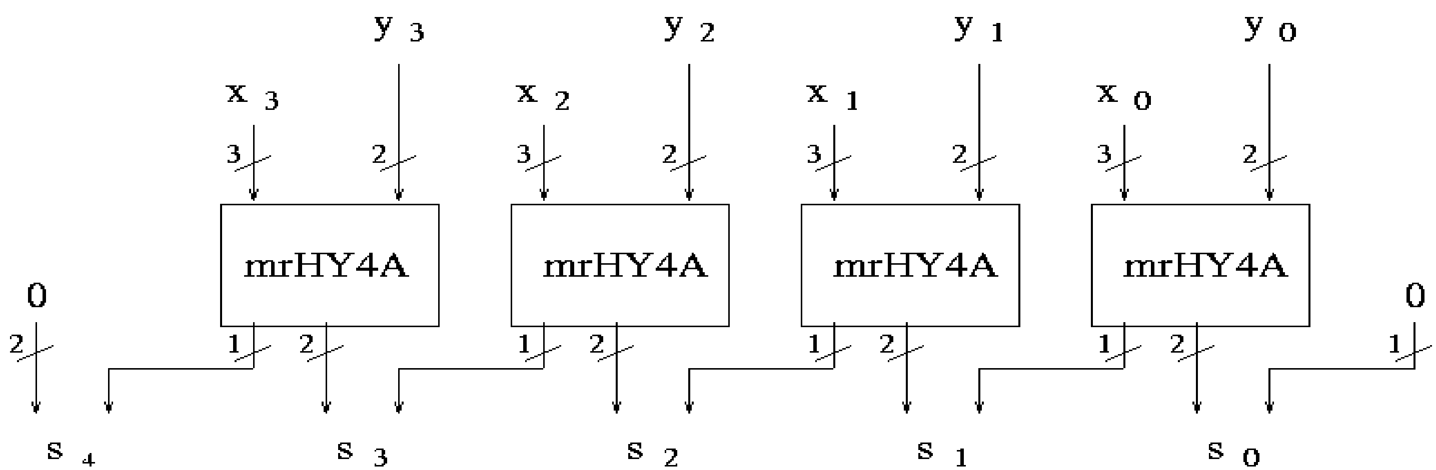
- Step 2 computes computes $s_i = t_{i-1} + u_i$. Replacing s_i, u_i, and t_{i-1} by corresponding binary codes leads to $s_i^{-2} = u_i^{-2}$, $s_i^{++} = t_{i-1}^{+}$ and $s_i^{+} = u_i^{+}$.

Digit	Radix 4 Digit Set	Binary Code
x_i	$\{\bar{2}, \bar{1}, 0, 1, 2\}$	$-2x_i^{-2} + x_i^{+} + x_i^{++}$
y_i	$\{0, 1, 2, 3\}$	$2y_i^{+2} + y_i^{+}$
$p_i = x_i + y_i$	$\{\bar{2}, \bar{1}, 0, 1, 2, 3, 4, 5\}$	$4t_i + u_i$
u_i	$\{\bar{2}, \bar{1}, 0, 1\}$	$2u_i^{+2} - 2u_i^{-2} - u_i^{-}$
t_i	$\{0, 1\}$	t_i^{+}
$s_i = u_i + t_{i-1}$	$\{\bar{2}, \bar{1}, 0, 1, 2\}$	$2s_i^{-2} + s_i^{+} + s_i^{++}$

Digit sets involved in Minimally Redundant Hybrid Radix-4 Addition



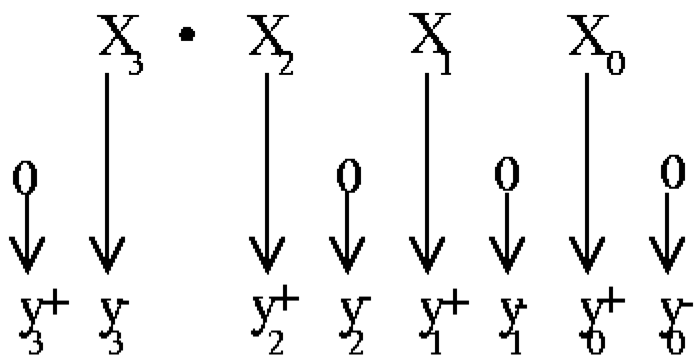
mrHY4A adder cell



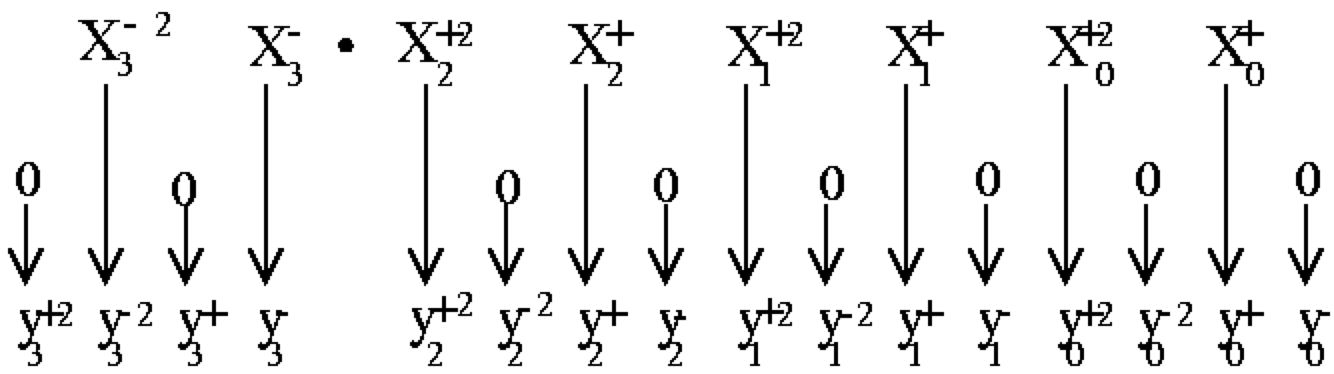
Four-digit mrHY4A

Non-redundant to Redundant Conversion

- Radix-2 Representation : A non-redundant number $X = x_3.x_2.x_1.x_0$ can be converted to a redundant number $Y = y_3.y_2.y_1.y_0$, where each digit y_i is encoded as y_i^+ and y_i^- as shown below:



- Radix-4 representation :
  - <u>radix-4 maximally redundant number</u>: X is a radix-4 complement number, whose digits x_i are encoded using 2 wires as $x_i = 2x_i^{+2} + x_i^{+}$. Its corresponding maximally redundant number Y is encoded using $y_i = 2y_i^{+2} - 2y_i^{-2} + y_i^{+} - y_i^{-}$. The sign digit x_3 can take values -3, -2, -1 or 0, and is encoded using $x_3 = -2x_3^{-2} - x_3^{-}$.



– <u>radix-4 minimally redundant number</u>: X is a radix-4 complement number, whose digits x_i are encoded using 2 wires as $x_i = 2x_i^{+2} + x_i^{+}$. Its corresponding minimally redundant number Y is encoded using $y_i = -2y_i^{-2} + y_i^{+} + y_i^{++}$. To convert radix-r number x to redundant number $y_{<r.\alpha>}$, the digits in the range $[\alpha, r - 1]$ are encoded using a transfer digit 1 and a corresponding digit $x_i - r$ where x_i is the i^{th} digit of x. Thus,

$$2x_i^{+2} + x_i^{+} = 4x_i^{+2} - 2x_i^{+2} + x_i^{+}$$
$$= y_{i+1}^{++} - 2y_i^{-2} + y_i^{+}$$